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ON THE GENERAL LINEAR DIFFERENTIAL EQUATION.

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Forsyth* gives the convergent series,

$$v = S_0 - S_1 + S_2 - \ldots,$$

as a general solution of the equation,

$$D^2 v + \mu v = 0;$$

wherein $S \equiv C + C'x$, and $S_k \equiv \int_0^x \int_0^x \mu\, S_{k-1}\, dx^2.$

An obvious generalization would be as follows:—

To solve the equation

$$D^n y + X_1 D^{n-1} y + \ldots + X_{n-1} Dy + X_n y = X, \qquad (1)$$

wherein $X_1, \ldots X_n$, X are functions of x only (or constants), let us operate upon (1_1)† with $D^{-n}, \equiv \int_0^x \int_0^x \ldots \int_0^x (\ )\, dx^n$; and let us operate upon (1_2) with $\int \int \ldots \int (\,)\, dx^n$, which differs from D^{-n} only by admitting n arbitrary constants of integration, $C_1, \ldots C_n$, and may be written

$$D^{-n}(\,) + C_1 . \frac{x^{n-1}}{(n-1)!} + \ldots + C_{n-1} . x + C_n .$$

Or if only a particular solution of (1) be desired, we may give convenient values to $C_1, \ldots C_n$; e. g. making $C_1 = \ldots = C_n = 0$, we may operate on both members of (1) with D^{-n}.

The first-named operators give

$$y + D^{-n} . X_1 D^{n-1} y + \ldots + D^{-n} . X_{n-1} Dy + D^{-n} . X_n y = \int \ldots \int X dx^n, \qquad (2)$$

Differential Equations, p. 109, Ex.

† By (1_1) and (1_2), respectively, I mean the first and second member of (1); by (3_{14}), the equation $(3_1) = (3_4)$; etc.

$$\therefore\ y = [1 + \{D^{-n}.X_1 D^{n-1}() + \ldots + D^{-n}.X_n()\}]^{-1}.\int\ldots\int X dx^n$$
$$= \int\ldots\int X dx^n - \{\ldots\}\int\ldots\int X dx^n + \{\ldots\}^2\int\ldots\int X dx^n - \ldots$$
$$= [1 - \{\ldots\} + \{\ldots\}^2 - \ldots]$$
$$\times [D^{-n}X + C_1\frac{x^{n-1}}{(n-1)!} + \ldots + C_{n-1}x + C_n]; \qquad (3)$$

wherein $\{\ldots\}[D^{-n}X + \ldots + C_n] \equiv D^{-n}.X_1 D^{n-1}[D^{-n}X + \ldots + C_n]$
$+ \ldots + D^{-n}.X_n[D^{-n}X + \ldots + C_n]$,

and $\{\ldots\}^k[D^{-n}X + \ldots + C_n] \equiv \{D^{-n}.X_1 D^{n-1}() + \ldots + D^{-n}.X_n()\}$
$\times \{\ldots\}^{k-1}[D^{-n}X + \ldots + C_n]$.

That (3_{14}) is a primitive of (1), appears by retracing the steps so as to get (1) from (3_{14}). Thus (1_1) is got from (2_1) and (1_2) from (2_2) by the same unideterminate operator D^n, to which both D^{-n} and $\int\ldots\int() dx^n$ are inverse; which, of course, justifies our having at first got (2_1) from (1_1) and (2_2) from (1_2) by different operators. Also, the primitive (3_{14}) is complete, having n effective arbitrary constants $C_1, \ldots C_n$.

To show the convergence of (3_4), let l be the length of the path followed by x in each of the n integrations implied in D^{-n}*; and let L_k, A, C_0 be positive real constants such that all along this path of x all the moduli of $\frac{l^k}{k!}, \frac{l^{k+1}}{(k+1)!}, \ldots$ are $< L_k$, and all those of $X_1, \ldots X_n$ are $< A$, and that of X is $< C_0$. Then, since no definite single integral can be larger† than the product of the length of its variable's path into the largest value taken by the function to be integrated, it will be seen that each term of (3_4), such as

$$\pm S_k, \equiv \pm\{\ldots\}^k[D^{-n}X + C_1\frac{x^{n-1}}{(n-1)!} + \ldots + C_{n-1}x + C_n],$$

is smaller than if all the integrations had been from $x = 0$ to $x = l$, and $X_1, \ldots X_n$ had each been constant and $= A$, and X had been the constant C_0. But in that case S_k would have reduced to

$$A^k(D^{-1} + \ldots + D^{-n})^k(C_0.D^{-n}1 + \ldots + C_{n-1}.D^{-1}1 + C_n),$$
$$< C_0.A^k n^k L_{k+n} + C_1.A^k n^k L_{k+n-1} + \ldots + C_{n-1}.A^k n^k L_{k+1} + C_n.A^k n^k L_k.$$

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*Unless it happen that the path must be lengthened to avoid regions where not all of $X_1, \ldots X_n$ X are holomorphic, l may be taken $=$ mod. x.

† If $N = A + Bi = R\varepsilon^{\theta i}$ and $N' = A' + B'i = R'\varepsilon^{\theta' i}$, wherein $i = \sqrt{-1}$ while A, A', B, B', R, R', θ, θ' are all real, and R, R' are not negative; then let us say that N is $\left\{\begin{matrix}\textit{greater}\\ \textit{less}\end{matrix}\right.$ than N', or $N \gtrless N'$, when $A \gtrless A'$; but that N is $\left\{\begin{matrix}\textit{larger}\\ \textit{smaller}\end{matrix}\right.$ than N' when $R \gtrless R'$. E.g., $-3 \pm i$ is larger but less than $2 \pm 2i$.

Hence (3_4) is not less convergent than

$$C_0 \cdot \sum_{k=1}^{k=\infty} A^k n^k L_{k+n} + \ . \ . \ . + C_n \cdot \sum_{k=1}^{k=\infty} A^k n^k L_k,$$

wherein the series multiplied by any constant C_r converges onward from $k = Aln - n + r$, slowly at first but finally very fast.

In the same way, if $X_1 = \ . \ . \ . = X_{m-1} = 0$, (3_4) is, at worst, convergent from $k = lm^{-1}[A(n-m+1)]^{1/m} - (n+1)m^{-1}$ onward, and the convergence will usually proceed about m times as fast as if $X_1 \ldots$ were not 0. To secure this advantage, we can, with Forsyth,* first transform (1) by the substitution $y = z\varepsilon^{-\frac{1}{n}\int X_1 \, dx}$ into $D^n z + P_2 D^{n-2} z + \ . \ . \ . + P_n z = P$; but the transformation may be unadvisable if . . . X_n, X be already in good form for repeated integration.

In estimating the final rate of convergence, A can safely be replaced by the largest of all the values assumed by mod. X_1, mod. $X_2^{\frac{1}{2}}$, . . . mod. $X_n^{1/n}$ during the integrations: e. g. if the series (3_4) is to be used only for values of x smaller than a value R, we need consider only the largest of the values of mod. X_1, . . . mod. $X_n^{1/n}$ when mod. $x <$ mod. R.

Forms of X_1, . . . X_n, X for which the successive integrations are rapid and produce terms that readily combine, will of course occur to the reader; but the only cases of much interest will be those where the law of the series (3_4) is clear.

The method may fail in the rare case where no path lies from 0 to x except across a line where not all of X_1, . . . X_n, X are finite.

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* Ibid.